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Binary Quantum Logic and Generating Semigroups

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A refined definition of basic concepts for logic describing physical systems is proposed. Within the suggested formalism of generating semigroups the active logic of questions and passive logic of answers are introduced. The objects for which both logics are isomorphic are called self-adequate. It is shown that the assumption of self-adequacy implies that the object is either quantum or classical. The possibility of application of the theory to non-self-adequate objects is discussed.

1. INTRODUCTION

The notion of quantum logic was first introduced by Birkhoff and von Neumann (1936). It was an attempt to give more evidence for the laws of quantum mechanics.

The traditional Copenhagen interpretation supposes that a property of a physical object is not inherent in the object but merely characterizes the procedure of measurement of the property. The quantum logic interpretation of quantum mechanics assumes that properties of an object are inherent in the object itself, but the laws of logic operating with the properties may differ from usual ones.

What does "a logic operating with properties" mean? We can always put in correspondence with any pair of properties a, b of an object their conjunction $a \wedge b$ (a and b) and disjunction $a \wedge b$ (a or b), which are also the properties of the object. Properties of a classical object are described by measurable subsets of its phase space, which form a Boolean algebra. Conjunctions of properties are described by intersections of corresponding subsets, and disjunctions correspond to set-theoretic union. Properties of a quantum object are described by closed subspaces of its Hilbert state

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space. Conjunctions correspond to intersection of subspaces, and disjunctions are described by closed linear spans of subspaces. The property lattice in this case is not Boolean: the law of distributivity does not hold. The detail investigation of axioms for the property lattice of quantum systems was carried out by Piron (1964).

In Section 2, following the traditional quantum logic approach (Jauch, 1968), I consider the property lattice as a background mathematical object describing a physical system. The main drawbacks of this approach are underlined: it is more cumbersome than the conventional one and looks merely like an algernative language on which one can formulate the results of classical and quantum mechanics.

In Section 3 the new background of generating semigroups for physical systems is introduced. The closure operator on this semigroup is defined and the collection of closed subsets is interpreted as the property lattice of the object.

In Section 4 I describe the logic generated by set-theoretic operations—the passive logic. It is usually called quantum logic of the object (Jauch, 1968).

In Section 5 I introduce the logic generated by the semigroup product—the active logic. This kind of logic has never been considered in physics. It was introduced by Girard (1987).

In Section 6 the connections between the two logics are discussed. The objects for which these logics are isomorphic are called self-adequate. The assumption of self-adequacy implies that the object is either classical or quantum (Appendix C).

Nonetheless, one has no reason to reject the possibility of the existence of objects which are *non-self-adequate.* Such hypothetical objects as a matter of principle cannot be described by quantum or by classical mechanics; however, the semigroup description stays valid.

Where could one look for non-self-adequate objects? The underlying mathematical concept of the formalism proposed is a semigroup. By each semigroup one can restore an automaton whose external behavior is described by this semigroup. Such an automaton can in turn be considered not only as an abstract one, but also as a model for some object. This object is not obliged to be tangible or material. It can be anything to which we are able to put questions and receive answers. However, the multiplicative and the additive logics describing such an object are not necessarily isomorphic. Moore (1956) considered a more special situation: a finite automaton which can be realized as a purely classical object displayed some features of quantum behavior. The further invesitgation of quantumlike automata was recently performed by Finkelstein and Finkelstein (1983) and Grib and Zapatrin (1989).

2. TRADITIONAL QUANTUM LOGIC: PHYSICAL OBJECT AS PROPERTY LATTICE

From this point on the objects we investigate are considered as a collection of all its properties. All the properties of an object form a lattice under conjunction and disjunction. If this lattice is *Boolean*, I call the object *classical*. If the lattice satisfies the *axioms of subspace lattice* (Piron, 1964), the object is called *quantum*. Under such an approach the notions of phase space or respectively state space become secondary and the essential difference between mathematics describing classical and quantum objects vanishes.

Besides the significant methodological advantages, the description of physical systems by means of their property lattices has a remarkable drawback. Lattices are not suitable mathematical objects to work with. One has to coordinate them, namely to represent them as algebras of subsets or subspaces, thus reverting to conventional description. To ensure that the description is universal so as to make no distinction between classical and quantum objects, we must find a universal way to represent the lattices.

Note that property lattices for both classical and quantum systems are complete. That means that there exist conjunctions and disjunctions for any number of elements. However, any complete lattice can be represented as an algebra of closed subsets of some set endowed with a closure operation. This construction is called polarity and was introduced in general form by Birkhoff (1967).

3. THE REPLACEMENT OF BACKGROUND: PHYSICAL OBJECT AS SEMIGROUP

First we describe the Birkhoff polarity construction. Consider a set P and a symmetric binary relation ψ on P. To any subset $A \subset P$ we can put into correspondence its polar $A^{\psi} \subset P$ defined as follows:

$$A^{\psi} = \{p \in P \mid \forall a \in A, p\psi a\} \tag{1}$$

Any subset $B \subset P$ of the form $B = A^{\psi}$ is called a *polar*. Polars for any symmetric relation always form a complete lattice $\Gamma_{\psi}(P)$. For any subset $A \subset P$ its *closure* Cl A is defined as bipolar:

$$\text{Cl}\, A = (A^{\psi})^{\psi}$$

The lattice $\Gamma_{\psi}(P)$ is then the lattice of closed subsets of P, namely the subsets satisfying the condition

$$A = \text{Cl}\, A$$

If the relation ψ is irreflexive,

$$a\psi a \;\Rightarrow\; \forall p \in P, a\psi p \tag{2}$$

then $\Gamma_\psi(P)$ is a complete lattice with orthocomplementation (1). To be distributive, $\Gamma_\psi(P)$ must also satisfy Šik's condition (Šik, 1981),

$$\forall p, q \in P, \quad p\bar{\psi}q \Rightarrow (\exists v \in P \mid r\bar{\psi}r \,\&\, r\psi p \,\&\, r\psi q) \tag{3}$$

The detailed consideration of the mathematical properties of polarities is beyond the scope of this paper. I only emphasize that classical objects with their distributive property lattices are merely special cases of more general systems specialized by conditions (2), (3).

The basic concept of the formalism I suggest is the *generating semigroup*, which is put into correspondance to a physical object. Closed subsets of the semigroup form a property lattice of the object. In detail, this construction looks as follows.

An abstract semigroup P is considered as a set of all conceivable elementary coercions (or questions, or experiments) on the object. The semigroup product $p \cdot q$ in P is interpreted as the joint effectuation of both coercions p and q. The semigroup P is not supposed commutative: in general $pq \neq qp$. Besides, P always contains a unit element $e \in P$ understood as "doing nothing."

The relation ψ is defined by fixing a subset $\psi \in P$ which is called the *absurd subset.* Elements of the absurd subset are interpreted as questions we are unable to put or experiments we are unable to perform. Define

$$p\psi q \overset{\text{Def}}{\Leftrightarrow} pq \in \psi \tag{4}$$

So $p\psi q$ means that coercions p and q are mutually exclusive. The request of symmetricity of the relation ψ restricts the possible choice of the subset ψ: it must be reflexive, i.e. (Thierrin, 1957):

$$\forall p, q \in P, \qquad pq \in \psi \Rightarrow qp \in \psi$$

Thus, if we set up a semigroup P and a reflexive subset $\psi \subset P$ we can unambiguously construct the lattice $\Gamma_\psi(P)$ which can be considered as a property lattice of some object. This lattice is called the passive or additive logic of the object.

4. ADDITIVE LOGIC

As mentioned above, any property is understood as the set of elementary coercions fitting this property. Let P be a generating semigroup with absurd set ψ and $A \subset P$ be a property. Which set corresponds to the negation of A? We define the negation of the property A as the set of all experiments mutually exclusive with any experiment fitting A. It is none other than the polar of A:

$$\neg A = \{p \in P \mid \forall a \in A, p\psi a\} = A^\psi \tag{5}$$

To keep the law of double negation valid, we must assume that for any property A

$$(A^{\psi})^{\psi} = A$$

Therefore properties of the object correspond to closed subsets of P. That is why $\Gamma_{\psi}(P)$ is called the property lattice for (P, ψ).

Meets in $\Gamma_{\psi}(P)$ are usual set-theoretic intersections: they are interpreted as conjunctions of properties. Joins in $\Gamma_{\psi}(P)$ are defined as follows:

$$A \vee B = \mathrm{Cl}(A \cup B)$$

The algebra $(\Gamma_4(P), \neg, \wedge, \vee)$ is called the *additive logic* of the object with generating semigroup (P, ψ).

In Appendix A the formalism of classical mechanics is converted in terms of generating semigroups. For quantum systems such a conversion is described in Appendix B. The additive logic of Appendix B is what is usually called a quantum logic.

5. MULTIPLICATIVE LOGIC

Using the semigroup product in P, we can construct one more logic for (P, ψ) generated by a product of subsets of P. The negation in this logic is defined in the same way as in the additive one. "Not A" is a polar A^{ψ}.

Let A, B be properties. The multiplicative conjunction $A \otimes B$ is defined as the set of all mutual coercions fitting A and B, respectively, namely

$$c \in A \otimes B \overset{\mathrm{Def}}{\Leftrightarrow} \exists a \in A, b \in B \mid c = ab$$

Thus

$$A \otimes B = \mathrm{Cl}(A \cdot B)$$

Multiplicative conjunction $A \otimes B$ is the least property containing all mutual coercions of the form $a \cdot b$.

The *multiplicative disjunction* $A \oplus B$ is defined using De Morgan's law:

$$A \oplus B = \neg(\neg A \otimes \neg B) = (A^{\psi} B^{\psi})^{\psi}$$

The algebra $(\Gamma_{\psi}(P), \neg, \otimes, \oplus)$ is called the *multiplicative logic* of the object with generating semigroup (P, ψ). For commutative semigroups the notion of multiplicative logic was introduced by Girard (1987).

In general, multiplicative logic has many undesirable features: sometimes $A \otimes A \neq A$, $A \otimes B \neq B \otimes A$, etc. However, as is shown in Appendices A and B for both classical and quantum objects the additive and the multiplictive logics are isomorphic.

One more special case treated in detail is when P is a semigroup with involution and ψ is an ideal in P satisfying certain conditions. Such a semigroup is called a Baer *-semigroup and the lattice $\Gamma_\psi(P)$ is isomorphic to a lattice of projectors in a Hilbert space or direct sum of Hilbert spaces (Pool, 1968).

6. CONNECTIONS BETWEEN THE LOGICS

The concept of property is identical in the two logics in both the substantial and mathematical senses. Substantially a property is the set of elementary coercions (elements of generating semigroup) fitting this property. Mathematically the property is described as a closed subset of P (an element of the property lattice). The difference between the two logics emerges in considering the semantics of logical operations.

Let A, B be properties. The additive conjunction $A \wedge B$ is their set-theoretic intersection. Each element of $A \wedge B$ fits *both* A and B. To fit $A \wedge B$ we have no need to perform any additional coercions on the object. Thus, the additive logic can be also called *passive*. It describes the logical connections between possible *answers* on the questions put to the system.

Each coercion fitting the multiplicative conjunction $A \otimes B$ is interpreted as a mutual performing of coercions fitting A and B, *respectively*. If we know which elements of P fit A and B, it is still not sufficient for finding all elements fitting $A \otimes B$. We must effectuate new coercions—mutual ones. So the multiplicative conjunction demands new experiments, and multiplicative logic thus can be called *active*. To perform an experiment is, in other words, to put a question to the system. That is why multiplicative logic can also be called the logic of *questions*.

In some special cases the two logics can be isomorphic, namely

$$\forall A, B \in \Gamma_\psi(P), \qquad A \wedge B = A \otimes B$$

Objects possessing such logics are called self-adequate. As was mentioned in Section 5, for both classical and quantum systems the additive logic is isomorphic to the multiplicative one. Thus, they are self-adequate. Moreover, as is shown in Appendix C, *only* these objects are self-adequate.

Remark. There are many mathematical connections between operations of additive and multiplicative logics. Some of them are treated by Girard (1987). However, the mathematical investigation exceeds the bounds of this paper.

8. SUMMARY

Here I put together the basic concepts of the formalism proposed.

1. We put into correspondance to any object its *generating semigroup* (P, ψ). Elements of P are understood as conceivable elementary coercions, or questions on the object.

2. The *absurd set* $\psi \subset P$ is a reflexive ($pq \in \psi \Rightarrow qp \in \psi$) subset of P. Elements of ψ are understood as nonperformable coercions.

3. Let $A \subset P$ be a set of questions. The *negation* of A is defined as a polar to the set A—a set of questions mutually exclusive with any question from A:

$$\neg A = \{ p \in P \mid \forall a \in A, pa \in \psi \} = A^{\psi}$$

4. A *property* of the object is any *closed* subset A, i.e., coinciding with its double negation: $A = \neg\neg A = (A^{\psi})^{\psi}$.

All the properties form a lattice denoted by $\Gamma_{\psi}(P)$. If $a \in A$, we say that the question a fits the property A.

5. The *additive conjunction* (passive AND) is defined on $\Gamma_{\psi}(P)$ as a set-theoretic intersection:

$$A \wedge B = A \cap B$$

The property $A \wedge B$ consists of all questions fitting *both* A and B.

6. The *multiplicative conjunction* (active AND) is defined as a closure of the semigroup product of properties:

$$A \otimes B = \mathrm{Cl}(AB)$$

$$AB = \{ ab \mid a \in A \;\&\; b \in B \}$$

The property $A \otimes B$ is the least property containing all questions of the form ab, where a, b fit A, B, *respectively*.

7. Both the additive and the multiplicative *disjunctions* are defined in accordance with De Morgan's law:

$$A \vee B = \neg(\neg A \wedge \neg B) = (A^{\psi} \cap B^{\psi})^{\psi}$$

$$A \oplus B = \neg(\neg A \otimes \neg B) = (A^{\psi} B^{\psi})^{\psi}$$

8. Thus, the pair (P, ψ) generates two algebras of sets $(\Gamma_{\psi}(P), \neg, \wedge, \vee)$ and $(\Gamma_{\psi}(P), \neg, \otimes, \oplus)$, which are called the additive and the multiplicative logic, respectively. The former is what is usually called the quantum logic of the system (Jauch, 1968; Finkelstein and Finkelstein, 1983).

9. An object is called *self-adequate* if its multiplicative and additive logics are isomorphic. Both classical and quantum objects, and only they, are self-adequate (see Appendix C).

10. The formalism proposed is applicable to any object to which we are able to put questions and receive answers. In the classical and quantum areas it gives nothing new above the conventional. Thus, the area of its possible applications is in a hypothetical domain of non-self-adequate objects.

APPENDIX A. CLASSICAL MECHANICS IN SEMIGROUP LANGUAGE

Let V be a phase space of a classical system S. Any observable on S can be interpreted as real-valued measurable function on V. Let P be the set of all observables. The product in P is defined pointwise:

$$\forall v \in V \qquad (fg)(v) = f(v) \cdot g(v)$$

The absurd subset ψ consists of the only zero observable. Let A be an arbitrary subset of P. Consider its polar:

$$A^{\psi} = \{p \in P \mid \forall a \in A, \forall v \in V, p(v) \cdot a(v) = 0\}$$

Evidently $p(v) \in A^{\psi}$ if its support supp p does not intersect with the support of any element of A. Thus A^{ψ} is a class of observables whose supports do not intersect with the union of supports of elements of A. This class can be characterized by certain subset of V, namely $V \setminus \bigcup_{a \in A} \operatorname{supp} a$. Therefore, any element of $\Gamma_{\psi}(P)$, since it has a form A^{ψ}, corresponds to the unique element of $B(V)$—an algebra of measurable subsets of V. Thus,

$$\Gamma_{\psi}(P) \simeq B(V)$$

The additive logic on $\Gamma_{\psi}(P)$ is isomorphic to the Boolean structure of $B(V)$. Here $a \in A$ means supp $a \subset A$. In order to define multiplicative conjunction, consider the product of two properties:

$$AB = \{ab \mid \operatorname{supp} a \subset A \;\&\; \operatorname{supp} b \subset B\}$$

$p \in AB$ iff supp p contains in both A and B, i.e., contains in their intersection. Thus $AB = A \cap B$, hence

$$A \otimes B = A \wedge B$$

Thus, the two logics are isomorphic.

APPENDIX B. QUANTUM MECHANICS IN SEMIGROUP LANGUAGE

Consider a quantum system with the state space $\mathscr{H}$. In this case P is the semigroup of bounded linear operators on $\mathscr{H}$ and ψ is the null operator. Let $A \subset P$. Consider the polar A^{ψ}. Here $b \in A^{\psi}$ means that for any $a \in A$, $ba = 0$. Thus A^{ψ} is the class of operators whose images contain in the intersection of kernels of operators from A. This class can be characterized by this intersection itself—a closed subspace of $\mathscr{H}$. Thus we can put into correspondance to any polar A^{ψ} the subspace of $\mathscr{H}$; hence

$$\Gamma_{\psi}(P) \simeq \mathscr{L}(\mathscr{H})$$

The additive logic of (P, ψ) is isomorphic to the lattice structure of $\mathcal{L}(\mathcal{H})$—the lattice of closed subspaces of $\mathcal{H}$. Here $a \in A$ means $A \subset \operatorname{Ker} \alpha$. Consider the product of two properties $A \cdot B$. Since *properties* A, B contain all operators with kernels containing the subspaces A, B, respectively, the kernel of any product operator $a \cdot b$ will contain the intersection $A \cap B$. Thus, $A \otimes B = A \cap B$ and, as in the classical case, the two logics are isomorphic, too.

Remark. The reasoning above is merely a sketch of a more rigorous investigation considering the algebras of left polars and right polars. However, a more detailed analysis would exceed the bounds of this paper.

APPENDIX C. CONSEQUENCES OF SELF-ADEQUACY

Suppose an object with generating semigroup (P, ψ) is self-adequate. This means that the algebras $(\Gamma_\psi(P), \neg, \wedge, \vee)$ and $(\Gamma_\psi(P), \neg, \otimes, \oplus)$ are isomorphic, or, in other words,

$$A \otimes B = A \cap B \qquad \text{(C1)}$$

for any A, $B \in \Gamma_\psi(P)$.

Let $A = \psi$, $B = P$. Then $\psi \otimes P = \operatorname{Cl}(\psi P) = \psi$; hence $\psi = \psi P$. Analogously, assigning P to A and ψ to B in (C1), we obtain $\psi = P\psi$. Thus, ψ is a two-sided ideal in P. Consider the Rees quotient semigroup $\tilde{P} = P/\psi$. The absurd set in $\tilde{P}$ will be O; however, $\Gamma_\psi(P) \simeq \Gamma_0(P)$. Since ψ is a reflexive ideal in P, $\tilde{P}$ is a Baer semigroup and $\Gamma_0(\tilde{P})$ is a product of weakly orthomodular lattices. This product can be imbedded into the property lattice of a physical system whose state space is an integral of Hilbert spaces. If the integral consists of the only lattice, it corresponds to a purely quantum system (Gudder *et al*, 1982). If all the lattices in the integral turn into two-element Boolean algebras, the property lattice $\Gamma_0(\tilde{P})$ is Boolean and thus describes the classical system. For finite systems this construction is described in detail by Grib and Zapatrin (1989).

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